A Surprising Connection between Two Proofs of the Infinitude of Primes

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Euclid and History
Euclid’s Proof of the Infinitude of Primes
Furstenberg’s Proof
Mercer’s Variation
A Connection

Euclid of Alexandria, 300BC
Euclid and History

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other non-Euclidean geometries emerged in the late 19th century.
one of the oldest surviving fragments of *The Elements*, 100AD
Euclid, depicted in Rafael’s *School of Athens* (1510)
The Infinitude of Primes

Let’s recall:

Definition

Note the number 1 is neither a prime nor composite. It is generally referred to as a *unit*. 
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**Definition**

- A **prime number** is a natural number greater than 1 that has no positive divisors other than 1 and itself.
- A **composite number** is a natural number greater than 1 that is not prime.

Note the number 1 is neither a prime nor composite. It is generally referred to as a *unit*. 
In Book 9 of *The Elements*, Euclid established the following.

**Main Theorem**

*There exists an infinite number of primes.*
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- and more.
Euclid’s Proof (300BC)

First we recall the all-important ...

**Fundamental Theorem of Arithmetic**

*For all* $n \in \mathbb{Z}$ *such that* $n > 1$, *$n$ can be represented uniquely as the product of primes.*

For example, the set $\{..., -11, -4, 3, 10, 17, 24, ...\}$ is an arithmetic progression, where $a = 7$ and $m = 3$. 
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Fundamental Theorem of Arithmetic

*For all* \( n \in \mathbb{Z} \) *such that* \( n > 1 \), \( n \) *can be represented uniquely as the product of primes.*

**Definition**

Let \( a, m \in \mathbb{Z} \). An arithmetic sequence is a set of integers of the form

\[
a + m\mathbb{Z} = \{a + mn : n \in \mathbb{Z}\}
\]

For example, the set \( \{..., -11, -4, 3, 10, 17, 24, ...\} \) is an arithmetic progression, where \( a = 7 \) and \( m = 3 \).
Lemma

For all integers \( m \) not equal to \(-1\) or \(1\),

\[ m\mathbb{Z} + 1 \subseteq \mathbb{Z} \setminus (m\mathbb{Z}). \]

I.e., one more than a multiple of \( m \) is not a multiple of \( m \).
Lemma

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Proof.

Let $mk + 1 \in m\mathbb{Z} + 1$ for some $k \in \mathbb{Z}$. Suppose by way of contradiction that $mk + 1$ is a multiple of $m$. Then there exists $n \in \mathbb{Z}$ such that $mk + 1 = mn$. Thus $1 = m(n - k)$ and $m$ divides 1. So $m$ must be either $-1$ or $1$, which is contradiction. We conclude $m\mathbb{Z} + 1 \subseteq \mathbb{Z} \setminus (m\mathbb{Z})$. \qed
Proof of the Infinitude of Primes (Euclid).

Let \( F = \{ p_1, \ldots, p_n \} \) be any finite list of primes. We show there is a prime not on our list \( F \).

Suppose that \( N = p_1 p_2 \ldots p_n + 1 \) were not prime. As \( N > 1 \), by the FTA there exists a prime \( p \) that divides \( N \).

If \( p \notin F \), then there is prime not on our list and we're done.

Otherwise, \( p \in F \) and note \( N \in p \mathbb{Z} + 1 \). As \( p > 1 \), it follows by the Lemma that \( N \in p \mathbb{Z} + 1 \subseteq \mathbb{Z} \setminus (p \mathbb{Z}) \).

This is contradiction since \( p \) divides \( N \). We conclude \( N \) is a prime, and furthermore it cannot be on our list \( F \).

Since for any finite list \( F \) of primes there is a prime not on our list, we conclude the set \( P \) of primes is infinite.
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- This is contradiction since $p$ divides $N$. We conclude $N$ is a prime, and furthermore it cannot be on our list $F$.
- Since for any finite list $F$ of primes there is a prime not on our list, we conclude the set $P$ of primes is infinite.
Hillel Furstenberg

A Connection between Two Proofs of the Infinitude of Primes
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currently at Hebrew University of Jerusalem, works in differential geometry and ergodic theory
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The Evenly-Spaced Integer Topology on $\mathbb{Z}$

**Definition**

The *evenly-spaced integer topology on* $\mathbb{Z}$ *consists of the following collection of open sets:*

$$\{ U \subseteq \mathbb{Z} : a\mathbb{Z} + b \subseteq U \text{ for some } a, b \in \mathbb{Z} \}.$$  

In other words, a non-empty set of integers is open in this space if and only if it contains an arithmetic sequence.
This amazing thing about this topology is that it actually is a topology! To help see why, let’s consider this question:

**Question**

*What can we say about the intersection of finitely many arithmetic sequences? That is, what are the possibilities for*

\[
\bigcap_{i=1}^{n} (a_i + m_i \mathbb{Z}),
\]

*where \(a_1, \ldots, a_n\) and \(m_1, \ldots, m_n\) are integers?*
We see that:

**Lemma**

*A finite intersection of arithmetic sequences is an arithmetic sequence (and thus infinite), or empty.*
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**Lemma**

A *finite intersection of arithmetic sequences is an arithmetic sequence (and thus infinite)*, or empty.

- It follows from the previous Lemma that the finite intersection of open sets is open in the Evenly-Spaced Integer Topology.
- Other conditions for a topology are also satisfied.
Two curious properties of this space:

1. an arithmetic sequence is both open and closed (clopen)
   Why?
Two curious properties of this space:

1. an arithmetic sequence is both open and closed (clopen)
   Why?

2. a finite set is not open (unless it is empty) as it cannot contain an infinite arithmetic sequence.
Proof (Furstenberg, 1955).

Suppose by way of contradiction that the set of primes were finite. By the Fundamental Theorem of Arithmetic, $\bigcup p_{\text{prime}} \mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}$. For each prime $p$, the arithmetic sequence $p \mathbb{Z}$ is closed. As the finite union of closed sets is closed, the set on the left above is closed. Hence, the set on the right is closed, implying $\{-1, 1\}$ is open. This is a contradiction since finite sets cannot be open.
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- Hence, the set on the right is closed, implying \( \{-1, 1\} \) is open.
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In 2009, Mercer “unpackaged” the topology in Furstenberg’s proof to uncover the underlying number theory. We give Mercer’s proof, also published in the *Monthly*.
If \( m \geq 2 \), then

\[
\mathbb{Z} \setminus (m\mathbb{Z}) = (1 + m\mathbb{Z}) \cup \ldots \cup ((m - 1) + m\mathbb{Z})
\]

I.e., \( \mathbb{Z} \setminus (m\mathbb{Z}) \) is a finite union of arithmetic sequences.
Proof of the Infinitude of Primes (Mercer’s Unpackaging).

Suppose that the set of primes were finite, and let $p_1, \ldots, p_n$ represent all the prime numbers. The Fundamental Theorem of Arithmetic implies that every integer other than 1 and $-1$ are multiples of some prime. Put another way, the numbers 1 and $-1$ are the only integers that are not multiples of any prime.

It follows that $\{ -1, 1 \} = \mathbb{Z} \setminus (p_1 \mathbb{Z}) \cap \mathbb{Z} \setminus (p_2 \mathbb{Z}) \cap \cdots \cap \mathbb{Z} \setminus (p_n \mathbb{Z})$.

Each $\mathbb{Z} \setminus (p_i \mathbb{Z})$ above is a finite union of arithmetic sequences, by the previous lemma. So $\{ -1, 1 \}$ is then a finite intersection of finite unions of arithmetic sequences.
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- it follows that

$$\{-1, 1\} = \mathbb{Z}\setminus(p_1\mathbb{Z}) \cap \mathbb{Z}\setminus(p_2\mathbb{Z}) \cap \cdots \cap \mathbb{Z}\setminus(p_n\mathbb{Z}).$$
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- Each $\mathbb{Z} \setminus (p_i \mathbb{Z})$ above is a finite union of arithmetic sequences, by the previous Lemma.
- So $\{ -1, 1 \}$ is then a finite intersection of finite unions of arithmetic sequences.
Proof, con’t.

finite intersections distribute over finite unions

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By a previous lemma, finite intersections of arithmetic sequences are empty or infinite, and so then will unions of these intersections.

thus \{-1, 1\} is either empty or infinite which, on most days of the week, it is decidedly not.

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A Connection Between the Proofs

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Suppose the set of primes $P$ where finite. Let $P = \{p_1, \ldots, p_n\}$.

Let $A \subseteq \mathbb{Z}$ be all the integers that are not multiples of any prime. Then,

$$A = \bigcap_{i=1}^{n} \mathbb{Z} \setminus (p_i \mathbb{Z}).$$
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The Fundamental Theorem of Arithmetic says that $A = \{-1, 1\}$. 

Observe that in Mercer's variation on Furstenberg's proof, the key idea is to show that $A$ is infinite, contradicting that $A = \{-1, 1\}$. (Thus there must be an infinitude of primes).
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- Observe that in Mercer’s variation on Furstenberg’s proof, the key idea is to show that $A$ is infinite, contradicting that $A = \{-1, 1\}$. (Thus there must be an infinitude of primes).
A straightforward way to see that $A$ is infinite (if the set of primes $P = \{p_1, \ldots p_n\}$ were finite):

- Let $m \in \mathbb{Z}$. 

- Notice that for any prime $p_i$, the product $mp_1p_2\cdots p_n$ is a multiple of $p_i$.
- By a previous Lemma, it follows that $mp_1p_2\cdots p_n + 1 \in \mathbb{Z} \setminus (p_i\mathbb{Z})$.
- So, $mp_1p_2\cdots p_n + 1 \in A$.
- Since the above holds for any $m \in \mathbb{Z}$, we see that $A$ is infinite.
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Now let’s go back and look at Euclid’s original proof. We see that

- A finite set of primes \( \{p_1, \ldots, p_n\} \) and the number
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\prod_{i=1}^{n} p_i + 1 \in \bigcap_{i=1}^{n} \mathbb{Z} \setminus (p_i \mathbb{Z}) = A.
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- But \( p_1 p_2 \cdots p_n + 1 > 1 \) and \( A = \{-1, 1\} \). This is a contradiction.
In summary, we see that both proofs are very similar, in the following way:

- Suppose the set of primes $P = \{p_1, \ldots, p_n\}$ were finite.
In summary, we see that both proofs are very similar, in the following way:

- Suppose the set of primes $P = \{p_1, \ldots, p_n\}$ were finite.
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In summary, we see that both proofs are very similar, in the following way:

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- Euclid’s proof is the observation that if $P$ were finite then $p_1p_2\cdots p_n + 1 \in A$
- Both observations contradict that $A = \{-1, 1\}$. 

Nathan Carlson 

A Connection between Two Proofs of the Infinitude of Primes

Thank you!